Strategized Optimization of Accumulating Pension Wealth Portfolio under the Constant Elasticity of Variance (CEV) Model

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Abstract

An optimal Pension investment strategy, under the Constant Elasticity of Variance (CEV) model is developed. The Pension Fund Investor (PFI) invested in both a risky asset (stock) and a riskless asset (Cash account), modeled with CEV process and constant interest rate, respectively. Here, the Pension Fund Administrator (PFA) considered and investigated the relevance/significance of extra stochastic contribution, during non-turbulent period, as a form of extra voluntary contribution, as provided by the Pension Reform Act of 2006, as amended. A constrained Pension Wealth optimization program was developed and transformed into a nonlinear partial differential equation, using the associated Hamilton Jacobi Bellman equation. The explicit solution of the constant relative risk aversion (CRRA) is obtained, using Legendre transform, dual theory, and change of variable methods. I presented and proved theorem on pension wealth investment strategy and the optimal utility function is also presented. It is established herein, with the optimal utility function that the extra stochastic contribution is minimally significant to the satisfaction of the PFI, due to its partial presence in the optimal utility function strategy.

Key words: *Strategized; Portfolio; CRRA; CEV; Accumulating Pension.*

1. Introduction

The Stock market which have witnessed a low investment returns, due to serious global economic downturn [1], hence have necessitated the continuous reviews of the various existing economic models [2-13] that bothers on investment strategies, and many more. Based on the structure of the Defined Contribution (DC) Pension Reform Scheme, the satisfaction of the PFI is predetermined by the level of investment returns, which is a function of the investment strategy, hence the need to continue reviewing all these many economic models becomes necessary [8]. As the trading economy is versed, so are the various players. Investments in the Stock market, for instance can come from different funds such as single/private investors, corporations, Government, employees of labor, and many more. In this work, our interest is in the later, as it is rooted in the contributory Pension scheme of 2006, as amended.

There are two basic types of Pension scheme; the defined benefit (DB) and the defined contrition (DC). In this research, we shall only base our work on the DC scheme. In DC Pension scheme, the employers are to pay a stipulated amount into the Pension retirement savings account (RSA) alongside with the employer. At retirement, the lump sum and the annuity is predetermined by the total investment yield (returns), and not fixed as in the DB scheme. The beauty of this scheme is that it is fully funded, because of the contributory nature.

Many articles have been published on optimal portfolio management, whose riskless and risky assets are modeled, using both Geometric Brownian Motion (GBM) process [3-5, 14-15] and Constant Elasticity of Variance (CEV) process [6, 8, 12, 16-17].

1.1. Research Motivation

[3] Introduced a new concept, "Extra Contribution", in the literature. In their work, they simply applied the provision of the Pension Reform Act of 2006, as amended, where the contributors are at liberty to make extra contribution to encourage extra investment returns. In their work, the investor chose a Constant Relative Risk Aversion (CRRA) Utility function. Their work revealed the need for the Pension Manager to increase the proportion of his wealth to be invested in Bond and Stock, and reduce the proportion that is invested in Cash. Continuing in that direction, [4] replicated the work of [3], but in their work, they considered "extra Stochastic" contribution. Their result is similar to that of [3]. It is important to note that both [3] and [4] followed the well-known GBM in modeling their Pension wealth process. However, [8,] used the well-known CEV in modeling their Pension wealth process. Significant results and observations were made, which includes the formulation and the proof of a theorem that shows that the elastic parameter must not be equal to one, amongst other findings, and this motivated this work. My approach is similar to that of [3,4,8], but our interest is to verify the significance or otherwise of extra stochastic contribution a CEV-generated Pension wealth. An additional assumption is made in the sequel.

1.2. Domain of Research and Some Preliminaries

The domain of this article is the complete probability space, (Ω, F, P), where Ω is a real space and P is a probability measure, $\{w_s(t), w_t(t)\}$ are two standard orthogonal Brownian motions, $\{F_t(t), F_s(t)\}$ are right continuous filtrations whose information are generated by the two standard Brownian motions $\{w_s(t), w_t(t)\}$, whose sources of uncertainties are respectively to the Stock market and time variance.

Considering a complete and frictionless financial market that is continuously open over the fixed time interval $[0, T]$, $T > 0$ (the retirement age). I assume that the market is composed of the risk-less asset (Bond), and risky asset (Stock).

2. The Pension Wealth Constrained Optimization Program

Under this session, we shall present and discuss the trading economy and formulate the optimization model for the trading/investment period.

2.1. The Financial Market

We shall consider a trading economy that is characterized by a riskless asset (Money in the Bank), and a risky asset (Stock). That is, the PFA's portfolio consists of just two assets.

Let the risk-less asset, C_t, say, at any positive time, t, evolve as;

$$dC_t = rC_t dt, \tag{2.1.1}$$

Where r represents constant rate of interest.

Next, we denote the price of the risky asset (stock) at any positive time, t, by S_t, as in [6], and [8], thus;

$$dS_t = \mu S_t dt + kS_t^{\beta+1} dW_t, \tag{2.1.2}$$

Where $\mu(\mu > r)$ represents the instantaneous rate of return on stock, β (β ≤ 0) is the elastic constant parameter, k is a constant, kS_t^{β} represents the instantaneous volatility.

Let $\{W_t; t \geq 0\}$ denote a standard Brownian motion, defined on a probability space, $(\Omega, \mathrm{F}, \mathrm{P})$ where $F = \{F_t\}$ is an augmented filtration generated by the Brownian motion.

2.2. The Assumptions of the Pension Wealth Constrained Optimization Program

Consistent with the Nigerian Pension Reform Act of 2006, as amended [8], we make the following assumptions

(a) The Pension Scheme accumulates wealth
(b) There are different categories of contributors
(c) The contributors will not willingly withdraw from the scheme
(d) The trading economy is turbulent-free.
(e) Extra stochastic contribution is used as the probable amortization fund
(f) Orthogonal relationship is considered between stock and time.

2.3. Formation of the Constrained Optimization Program

Let us denote the investment made in Stock as u_s such that that kept as Money in the account is given by $u_c = 1 - u_s$.

Let the stochastic differential equation that governs the variance of wealth generation be given by

$$dy(t) = u_s y(t)\frac{ds_t}{s_t} + (1-u_s)y(t)\frac{dc_t}{c_t} + dp \tag{2.3.1}$$

Subject to:
$$\begin{aligned} &u_s > 0 \\ &u_c = 1 - u_s > 0 \end{aligned} \tag{2.3.1a}$$

Where;

$y(t)$ is the accumulated Pension wealth with time, t

dp is the regular contribution process, and is defined by;

$$dp = (1+\vartheta_i)\delta_{i+1}dt + \sigma dw_t \; and \; \sigma = ks^{\beta}, \tag{2.3.1b}$$
$$i = 4, 5.6, ..., n-1 \, and \; \vartheta_4 = 4, \vartheta_5 = 5, \vartheta_6 = 6, ..., \vartheta_n = n, \vartheta_1 > 0, an \operatorname{int} eger\,(staff\ loading)$$

Combining (2.1.1), (2.1.2), (2.3.1b) with (2.3.1) s.t (2.3.1a), and simplifying

$$dy(t) = y(t)\left[u_s\mu + r - u_s r\right] + (1+\vartheta_i)\delta_{i+1}dt + (u_s y(t) + 1)ks_t^{\beta}dw_t \tag{2.3.2}$$

Subject to:
$$\begin{matrix} u_s > 0 \\ u_c = 1 - u_s > 0 \end{matrix}, \tag{2.3.2a}$$

$$i = 4, 5.6, ..., n-1 \, and \; \vartheta_4 = 4, \vartheta_5 = 5, \vartheta_6 = 6, ..., \vartheta_n = n, \vartheta_1 > 0, an \operatorname{int} eger\,(staff\ loading),$$
$$\delta_i \; is \operatorname{var} ious\ amount\ contributed$$

Based on the wealth process in (2.3.2) s.t (2.3.2a), the PFA seeks a strategy, u_s^*, which maximizes the utility function, such that $u_s^* = \max E\left(U\left(Y(\mathrm{T})\right)\right), \forall u(\mathrm{t})$. Where $\mathrm{u}(\bullet)$ is an increasing concave utility function, which satisfies the Inada conditions;

$U'(+\infty) = 0, \text{and}\, U'(0) + \infty$ (cf. Gao [6])

3. The Pension Wealth Optimization

Applying the associated H. J. B. Equation to equation (2.3.2) s.t (2.3.2a), we derive

$$\begin{aligned} &H_t + H_s\mu s_t + H_y\left(y(t)\left[u_s\mu + r - u_s r\right] + \left[1+\vartheta_i\right]\delta_{i+1}\right) + \frac{1}{2}H_{ss}k^2 s_t^{2\beta+2} \\ &+ \frac{1}{2}H\left(u_s y(t) + u_s\right)^2 k^2 s_t^{2\beta} + H_{sy}k^2 s_t^{2\beta+1}\left(u_s y(t) + 1\right) = 0 \end{aligned} \tag{3.1}$$

Subject to:
$$\begin{matrix} u_s > 0 \\ u_c = 1 - u_s > 0 \end{matrix}, \tag{3.1a}$$

$$i = 4, 5.6, ..., n-1 \, and \; \vartheta_4 = 4, \vartheta_5 = 5, \vartheta_6 = 6, ..., \vartheta_n = n, \vartheta_1 > 0, an \operatorname{int} eger\,(staff\ loading),$$
$$\delta_i \; is \operatorname{var} ious\ amount\ contributed$$

To obtain the optimal value, u_s^*, we differentiate equation (3.1) s.t (3.1a) with respect to u_s, thus

$$
\begin{aligned}
&H_t + H_s \mu s_t + \frac{1}{2} H_{ss} k^2 s_t^{2\beta+2} + H_y \left(1+\vartheta_i\right) \delta_{i+1} + H_{sy} k^2 s_t^{2\beta+1} + \\
&\max\left(H_y y(t) \left(u_s \mu + r - u_s r\right) + \frac{1}{2} H_{yy} \left(u_s y(t)+1\right)^2 k^2 s_t^{2\beta} + H_{sy} k^2 s_t^{2\beta+1} u_s y(t) \right) = 0
\end{aligned} \tag{3.2}
$$

$$
\text{Subject to:} \quad \begin{aligned} &u_s > 0 \\ &u_c = 1 - u_s > 0 \end{aligned}, \tag{3.2a}
$$

$i = 4, 5.6, ..., n-1 \, and \, \vartheta_4 = 4, \vartheta_5 = 5, \vartheta_6 = 6, ..., \vartheta_n = n, \vartheta_1 > 0, an \operatorname{int} eger \, (staff \;\, loading),$
$\delta_i \; is \operatorname{var} ious \; amount \; contributed$

and this yields

$$
u_s^* = -\left(\frac{1}{y(t)} + \frac{(\mu - r)}{y(t) k^2 s_t^{2\beta}} - \frac{H_y}{H_{yy}} + \frac{s}{y(t)} \frac{H_{sy}}{H_{yy}} \right) \tag{3.2b}
$$

Replacing u_s in equation (3.2) with u_s^* in equation (3.2b)

$$
\begin{aligned}
&H_t + H_s \mu s_t + \frac{1}{2} H_{ss} k^2 s^{\beta+2} + H_y \left[(1+\vartheta_i) \delta_{i+1} - \mu + y(t) r + r \right] \\
&+ \frac{H_y^2}{H_{yy}} \left[-\frac{\mu^2 - r^2}{k^2 s_t^{2\beta}} + \frac{1}{2} \frac{\left(\mu - r\right)^2}{k^2 s_t^{4\beta}} \right] + s \frac{H_{sy} H_y}{H_{yy}} \left[-\mu + r \right] + \frac{s^2 H_{sy}^2}{H_{yy}} \left[-\frac{1}{2} k^2 s_t^{2\beta} \right] = 0
\end{aligned} \tag{3.3}
$$

$$
\text{Subject to:} \quad \begin{aligned} &u_s > 0 \\ &u_c = 1 - u_s > 0 \end{aligned}, \tag{3.3a}
$$

$i = 4, 5.6, ..., n-1 \, and \, \vartheta_4 = 4, \vartheta_5 = 5, \vartheta_6 = 6, ..., \vartheta_n = n, \vartheta_1 > 0, an \operatorname{int} eger \, (staff \;\, loading),$
$\delta_i \; is \operatorname{var} ious \; amount \; contributed$

Since the stochastic constrained control problem described in the previous session has been converted to a constrained nonlinear stochastic partial differential equation, next we solve for H in (3.3) s.t (3.3a) and subsequently substitute it into (3.2b), to enable us obtain the optimal wealth generation policy (i.e., the control strategy). To achieve this, we apply the Dual theory and Legendre transformation techniques, respectively.

4. The Dual and Legendre Transformation on (3.3) s.t (3.3a)

Here, we transform the nonlinear second order partial differential equation (4.4) into a linear PDE, using the Dual theory and Legendre transformations, not limited to Gao [6], Njoku et al [8], Zhang et al [15], that is;

$$\phi_y = z \text{ and } \phi_t = \hat{\phi}_t, \phi_s = \hat{\phi}_s, \phi_{ss} = \hat{\phi}_{ss} - \frac{\hat{\phi}_{sz}^2}{\hat{\phi}_{zz}}, \phi_{yy} = \frac{-1}{\hat{\phi}_{zz}}, \phi_{ys} = \frac{-\hat{\phi}_{sz}}{\hat{\phi}_{zz}}. \tag{4.1}$$

Taking into equations (4.1) and (3.3) s.t (3.3a)

$$\begin{aligned}&\widehat{H}_t + \widehat{H}_s \mu s_t + \frac{k^2 s^{\beta+2}\widehat{H}_{ss}}{2} - \frac{k^2 s^{\beta+2}\widehat{H}_{sz}^2}{2\widehat{H}_{zz}} + z(1+\vartheta_i) - z\mu + zy(t)r + zr + \frac{z^2\widehat{H}_{zz}^2\mu^2 + r^2}{k^2 s_t^{2\beta}} \\ &- \frac{z^2\widehat{H}_{zz}\left(\mu^2 - 2\mu r + r^2\right)}{2k^4 s_t^{4\beta}} - sz\mu\widehat{H}_{sz} + szr\widehat{H}_{sz} - \frac{s^2\widehat{H}_{sz}^2 k^2 s_t^{2\beta}}{2\widehat{H}_{zz}} = 0\end{aligned} \tag{4.2}$$

Setting $y = \theta = -\widehat{H}_z$ into (4.2), and differentiating the dual θ and value function H with respect to z, we obtain

$$\begin{aligned}&\theta_t + \mu s_t \theta_t + \frac{k^2 s^{\beta+2}\theta_{ss}}{2} - \frac{k^2 s^{\beta+2}\theta_z\theta_s\theta_{sz}}{\theta_z^2} + \frac{k^2 s^{\beta+2}\theta_s^2\theta_{zz}}{2\theta_z^2}(1+\vartheta_i) - \mu - (z\theta_z - \theta)r - r + \frac{z^2\mu^2\theta_{zz}}{k^2 s_t^{2\beta}} + \frac{2z\mu^2\theta_z}{k^2 s_t^{2\beta}} \\ &+ \frac{(z^2\theta_{zz} + 2z\theta_s)\left(\mu^2 - 2\mu r + r^2\right)}{2k^4 s_t^{4\beta}} + sz\mu z\theta_{zs} + s\mu\theta_s - szr\theta_{sz} + \frac{2s^2k^2 s_t^{2\beta}\theta_z\theta_s\theta_{zs}}{2\theta_z^2} - \frac{s^2k^2 s_t^{2\beta}\theta_s^2\theta_{zz}}{2\theta_z^2} = 0\end{aligned}$$

(4.3)

$$\text{Subject to: } \begin{aligned}&u_s > 0 \\ &u_c = 1 - u_s > 0\end{aligned}, \tag{4.3a}$$

$i = 4, 5.6, \ldots, n-1$ *and* $\vartheta_4 = 4, \vartheta_5 = 5, \vartheta_6 = 6, \ldots, \vartheta_n = n, \vartheta_1 > 0$, *an integer (staff loading)*,
δ_i *is various amount contributed*

where the associated strategy is given by

$$u_s^* = \frac{-1}{y(t)}\left(1 + \frac{(\mu - r)}{k^2 s_t^{2\beta}}\right) z\theta_z + \frac{s}{y(t)}\theta_s \tag{4.4}$$

5. Utility Function Test

To obtain the level of satisfaction the plan member gets from his/her investment, we obtain the explicit solution for the CRRA utility functions, using change of variable technique.

5.1 Explicit solution to the CRRA utility

Similar to Gao [6], Njoku et al [8] and Zhang et al [5], we use

$$\theta(t,s,z) = z^{\frac{1}{q-1}}, q < 1, q \neq 0 \tag{5.1.1}$$

Assuming a solution form to (4.3) s.t. (4.3a), we have

$$\theta(t,s,z) = z^{\frac{1}{q-1}} h(t,s) + \alpha(t); \alpha(T) = 0, h(T,s) = 1 \tag{5.1.2}$$

Obtaining the various first and second partial derivatives with respect to t, s, z, we have

$$\theta_t = h_t z^{\frac{1}{q-1}} + \alpha'(t); \theta_s = h_s z^{\frac{1}{q-1}}; \theta_z = \frac{-h}{1-q} z^{\frac{1}{q-1}-1}; \theta_{ss} = h_{ss} z^{\frac{1}{q-1}}; \theta_{sz} = \frac{-h}{1-q} z^{\frac{1}{q-1}-1}; \theta_{zz} = \frac{(2-q)h}{(1-q)^2} z^{\frac{1}{q-1}-2} = 0 \tag{5.1.3}$$

Taking into (5.1.1), (5.1.2), (5.1.3) and (4.3) s.t (4.3a), we have

$$z^{\frac{1}{q-1}}\left(\theta_t + \mu s_t \theta_t + \frac{k^2 s^{\beta+2}\theta_{ss}}{2} - k^2 s^{\beta+2}\theta_s + r\theta\left[\frac{2-q}{1-q}\right] - \frac{\mu^2 2\theta}{k s_t^{\beta}(1-q)} + \frac{s\mu\theta}{1-q} - s\mu\theta_s - \frac{sr\theta}{1-q} + s^2 k^2 s_t^{2\beta}\theta_s\right) +$$
$$z^{\frac{1}{q-1}+1}\left(\frac{\mu^2\theta_s}{k^4 s_t^{4\beta}} + \frac{2\mu r\theta_s}{k^4 s_t^{4\beta}} + \frac{r^2\theta_s}{k^4 s_t^{4\beta}}\right) + \left(\alpha'(t) + \mu s_t \alpha'(t) + \mu s_t \alpha'(t) - (1+\theta_i) - c_{i+1} + \mu - r\alpha\right) = 0$$

(5.1.4)

Factoring out terms that depends on $z^{\frac{1}{q-1}}$, and z, and the ones that is independent of either of the two mentioned, we split (5.1.4) into three, thus

$$z^{\frac{1}{q-1}}\left(\theta_t + \mu s_t \theta_t + \frac{k^2 s^{\beta+2}\theta_{ss}}{2} - k^2 s^{\beta+2}\theta_s + r\theta\left[\frac{2-q}{1-q}\right] - \frac{2\mu^2\theta}{k s_t^{\beta}(1-q)} + \frac{s\mu\theta}{1-q} - s\mu\theta_s - \frac{sr\theta}{1-q} - s^2 k^2 s_t^{2\beta}\theta_t\right) = 0$$

(5.1.5)

$$z^{\frac{1}{q-1}+1}\left(\frac{\mu^2\theta_s}{k^4 s_t^{4\beta}} + \frac{2\mu r\theta_s}{k^4 s_t^{4\beta}} + \frac{r^2\theta_s}{k^4 s_t^{4\beta}}\right) = 0 \tag{5.1.6}$$

$$\left(\alpha'(t) + \mu s_t \alpha'(t) + (1+\theta_i) - c_{i+1} + \mu - r\alpha\right) = 0 \tag{5.1.7}$$

This implies that

$$\left(\theta_t + \mu s_t \theta_t + \frac{k^2 s^{\beta+2}\theta_{ss}}{2} - k^2 s^{\beta+2}\theta_s + r\theta\left[\frac{2-q}{1-q}\right] - \frac{2\mu^2\theta}{ks_t^{\beta}(1-q)} + \frac{s\mu\theta}{1-q} - s\mu\theta_s - \frac{sr\theta}{1-q} - s^2k^2s_t^{2\beta}\theta_t\right) = 0$$

(5.1.8)

$$\left(\frac{\mu^2\theta_s}{k^4 s_t^{4\beta}} + \frac{2\mu r\theta_s}{k^4 s_t^{4\beta}} + \frac{r^2\theta_s}{k^4 s_t^{4\beta}}\right) = 0 \quad (5.1.9)$$

$$\left(\alpha'(t) + \mu s_t \alpha'(t) + (1+\theta_i) - c_{i+1} + \mu - r\alpha\right) = 0 \quad (5.1.10)$$

Solving (5.1.10) at the boundary condition, $\alpha(T) = 0$, we obtain the continuous annuity of duration, $T-t$, yields

$$\alpha(t) = -\frac{\left((1+\vartheta_i) - c_{i+1} + \mu\right)(1+\mu s_t)}{r(1+\mu s_t)} \quad (5.1.11)$$

$$\mu = -r \pm \frac{r\sqrt{8}}{2} \quad (5.1.12)$$

Solving (5.1.8), observe firstly that the equation contains some variable coefficients, $s, s^{\beta+2}, s^2, s^{2\beta}$, and this makes obtaining solution somewhat difficult. However, in order to overcome this difficulty, we employ the services of power transformation and change of variable technique as in Gao [6], Zhang [15,], not limited to them.

Assuming,

$$h(t,s) = f(t,j),\ j = s_t^{2\beta} \quad (5.1.13)$$

Such that

$$h_t = f_t; h_s = 2\beta s^{(2\beta-1)} f_j; h_{ss} = 2\beta(2\beta-1)s_t^{2(\beta-1)} f_j + 2\beta s^{2(\beta-1)} f_{jj} \quad (5.1.14)$$

then putting (5.1.13) and (5.1.14) into (5.1.18), assuming that the elastic parameter, $\beta = 0$ (the GBM case) and simplifying gives

$$f_t\left(1 + \mu s_t f_t - s^2k^2 f_t\right) + f\left(r\left(\frac{2-q}{1-q}\right) - \frac{2\mu^2}{k(1-q)} + \frac{s\mu}{1-q} - \frac{sr}{1-q}\right) = 0 \quad (5.1.15)$$

Solving (5.1.15), by assuming that;

$f(t,j) = A_t \ell^{\beta(t)j};\ A(T)=1,\ \beta(T)=0 \Rightarrow f_t = A\ell^{\beta(t)j} + A(t)\beta_t j\ell^{\beta(t)j}$ and multiplying the outcome with $\frac{1}{A(t)}$, yields

$$\frac{A_t\ell^{\beta(t)j}}{A(t)} + \beta_t j\ell^{\beta(t)j} + \frac{\mu s_t A_t \ell^{\beta(t)j}}{A(t)} + \mu s_t \beta_t j\ell^{\beta(t)j} - \frac{s^2k^2A_t\ell^{\beta(t)j}}{A(t)}$$

$$-s^2k^2\beta_t j\ell^{\beta(t)j} + r\left(\frac{2-q}{1-q}\right)\ell^{\beta(t)j} - \frac{2\mu\ell^{\beta(t)j}}{k(1-q)} + \frac{s\mu\ell^{\beta(t)j}}{1-q} - \frac{sr\ell^{\beta(t)j}}{1-q} = 0 \quad (5.1.16)$$

$, j = s_t^{2\beta}$

Splitting (5.1.16) based on its dependency on $\ell^{\beta(t)j}$ and $\beta_t j\ell^{\beta(t)j}$, yields

$$\frac{A_t}{A(t)}\left(1+\mu s_t - s^2k^2\right) + r\left(\frac{2-q}{1-q}\right) - \frac{2\mu^2}{k(1-q)} + \frac{s\mu}{1-q} - \frac{sr}{1-q} = 0 \quad (5.1.17)$$

and

$$\beta_t\left(1+\mu s_t + s^2k^2\right) = 0 \quad (5.1.18)$$

In (5.1.18), $\beta_t = 0 \Rightarrow \int \beta_t = cons\tan t$

Considering terminal investment; $t=T,\ \beta(T)=0 \Rightarrow \beta(T) = cons\tan t \Rightarrow (cons\tan t = 0)$

Solving (5.1.18) by multiplying with $A(t)$ and dividing by $1+\mu s_t - s^2k^2$ and simplifying, yields

$$A_t + \frac{\left(r\left(\frac{2-q}{1-q}\right) - \frac{2\mu^2}{k(1-q)} + \frac{s(\mu-r)}{1-q}\right)}{\left(1+\mu s_t - s^2k^2\right)}A(t) = 0 \quad (5.1.19)$$

Solving the first order linear homogeneous equation (5.1.19), using integrating factor method

Let $I.F = \ell^{\int p(t)dt}$, where $p(t) = \frac{\left(r\left(\frac{2-q}{1-q}\right) - \frac{2\mu^2}{k(1-q)} + \frac{s(\mu-r)}{1-q}\right)}{\left(1+\mu s_t - s^2k^2\right)}$

Simplifying $p(t)$, we have

$$p(t) = \left(\frac{2rk - qrk - 2\mu^2 + s\mu k - srk}{(1-q)\left(1+\mu s_t - s^2k^2\right)}\right)$$

Consequently,

$$I.F=\ell^{\int\left(\frac{2rk-qrk-2\mu^2+s\mu k-srk}{(1-q)\left(1+\mu s_t-s^2k^2\right)}\right)dt} \tag{5.1.20}$$

Multiplying (5.1.19) with (5.1.20) and simplifying, we obtain

$$\frac{d}{dt}\left(\ell^{\left(\frac{2rk-qrk-2\mu^2+s\mu k-srk}{(1-q)\left(1+\mu s_t-s^2k^2\right)}\right)t}.A_t\right)=0 \tag{5.1.21}$$

Integrating (5.1.21) with respect to t, we have

$$A(t)=C\ell^{-\left(\frac{2rk-qrk-2\mu^2+s\mu k-srk}{(1-q)\left(1+\mu s_t-s^2k^2\right)}\right)t} \tag{5.1.22}$$

But,

$$h(t,s)=f(t,j),\ j=s_t^{2\beta} \tag{5.1.23}$$

and

$$\alpha(t)=-\frac{\left((1+\vartheta_i)-c_{i+1}+\mu\right)(1+\mu s_t)}{r(1+\mu s_t)} \tag{5.1.24}$$

Recall that

$$f(t,j)=A_t\ell^{\beta(t)j} \tag{5.1.25}$$

Therefore,

$$f(t,j)=C\ell^{-\left(\frac{2rk-qrk-2\mu^2+s\mu k-srk}{(1-q)\left(1+\mu s_t-s^2k^2\right)}\right)t}\ell^{\beta(t)j} \tag{5.1.26}$$

Recall also,

$$\theta(t,s,z)=z^{\frac{1}{q-1}}h(t,s)+\alpha(t);\alpha(T)=0,h(T,s)=1 \tag{5.1.27}$$

Theorem 1. By equations (5.1.27), (5.1.24), (5.1.23), and (5.1.26), the optimal stock investment strategy is given by

$$u_s^* = \frac{1}{y(t)}\left(1 - \frac{r\left(2+\sqrt{2}\right)}{k^2 s_t^{2\beta}}\right)\frac{z^{\frac{1}{q-1}}}{q-1} c\ell^{\left[\frac{2rk-qrk-2\left(-r-r\sqrt{2}\right)k-srk}{(1-q)\left(1+\left(-r-r\sqrt{2}\right)s_t - s_t^2 k^2\right)}\right]t}$$

Proof

Taking into (5.1.22), (5.1.23), (5.1.25) and (5.1.27)

$$\theta(t,s,z) = \frac{1}{z^{q-1}} c\ell^{-\left(\frac{2rk-qrk-2\mu^2+s\mu k-srk}{(1-q)\left(1+\mu s-s^2k^2\right)}\right)t} + \frac{\left((1+\vartheta_i) - C_{i+1} + \mu\right)(1+\mu s_t)}{r(1+\mu s_t)} \tag{5.1.28}$$

Differentiating (5.1.28) with respect to $s \text{ and } z$

$$\theta_s = 0 \tag{5.1.29}$$

and

$$\theta_z = \frac{1}{q-1} z^{\frac{1}{q-1}-1} c\ell^{-\left(\frac{2rk-qrk-2\mu^2+s\mu k-srk}{(1-q)\left(1+\mu s-s^2k^2\right)}\right)t} \tag{5.1.30}$$

Therefore, taking into (5.1.30), (5.1.29), (5.1.12) and (4.4)

$$u_s^* = \frac{1}{y(t)}\left(1 - \frac{r\left(2+\sqrt{2}\right)}{k^2 s_t^{2\beta}}\right)\frac{z^{\frac{1}{q-1}}}{q-1} c\ell^{\left[\frac{2rk-qrk-2\left(-r-r\sqrt{2}\right)k-srk}{(1-q)\left(1+\left(-r-r\sqrt{2}\right)s_t - s_t^2 k^2\right)}\right]t} \tag{5.1.31}$$

where,

$$\mu = -r \pm r\sqrt{2} : \mu_1 = -r - r\sqrt{2},\ \mu_2 = -r + r\sqrt{2} \qquad \square \tag{5.1.31a}$$

Consequently, our C.R.R.A utility function, $\theta(t,s,z)$ is given by taking into (5.1.27), (5.1.23), (5.1.26) and (5.1.28), thus

$$g(t,s,z) = z^{\frac{1}{p-1}}.ce^{-\left\{\frac{2\gamma k - p\gamma k - 2\mu^2 + s\gamma k - s\gamma k}{(1-p)\left(1+\mu s_t - s^2 k^2\right)}\right\}} + \frac{\left[(1+\theta_i) - c_{i+1} + \mu\right](1+\mu s_t)}{\gamma(1+\mu s_t)}$$

And this is the C.R.A.A utility function we sort.

6. Result and Discussion

It was observed that the extra stochastic contribution term partially appeared in the utility function, but the coefficient did, which does not really guarantee the significance or otherwise to the satisfaction of the PFI, considering the assumption of this model.

7. Conclusion

We studied and constructed pension wealth investment strategy in a defined contribution pension scheme, with more than one contributor. We developed a formula for wealth investment into stock, using C.R.R.A utility function. We also developed associated utility function. Based on our discovery, we therefore cannot strongly conclude that the stochastic extra contribution as adopted by the Pension Fund Investor contributes to the satisfaction of the Pension Fund Member.

8. Recommendation

Sensitivity analysis should be done on the optimal strategy and the Optimal utility function, and shown pictorially, in order to strongly conclude the relevance of the extra stochastic voluntary contribution, and as well, test for sensitivity of some existing parameters in the both the strategy and the utility function.

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